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Effect of thermal modulation on the onset of double diffusive convection in a horizontal fluid layer

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Abstract

The effect of time periodic boundary temperatures on the onset of double diffusive convection in a horizontal two component fluid layer is studied using a linear stability analysis. The perturbation method is used to compute the critical thermal Rayleigh number and the corresponding wave number for small amplitude temperature modulation. The correction thermal Rayleigh number is calculated as a function of frequency of the modulation, Prandtl number, solute Rayleigh number, and the diffusivity ratio. It is found that the thermal modulation may stabilize an unstable system or destabilize a stable system. In particular it is found that low frequency symmetric modulation is destabilizing whereas the asymmetric modulation and lower wall temperature modulation are stabilizing. The effect of the solute Rayleigh number, ratio of the diffusivities and the Prandtl number are also reported. It is also found that the effect of modulation disappears for large frequency.

Keywords: Double diffusion; Convection; Thermal modulation; Stability

1. Introduction

The study of convective motions produced by unstable density distributions in a fluid is now highly developed. Most attention has been given to the linear and nonlinear stability of a horizontal fluid layer heated from below and cooled from above. It has been shown that if gradients of two stratifying agencies, such as heat and salt, having different diffusivities are simultaneously present in a fluid layer, a variety of interesting convective phenomena can occur which are not possible in a single component fluid. The case of two or more stratifying agencies has been the subject of extensive theoretical and experimental investigations. Excellent reviews of these studies have been reported by Turner [1–3], Huppert and Turner [4], Platten and Legros [5]. The interest in the study of two or multicomponent convection has developed as a result of the marked difference between single component and multicomponent systems. In contrast to single component systems, convection sets in even when density decreases with height, that is, when the basic state is hydrostatically stable. The double diffusive convection is of importance in various fields such as high quality crystal production, oceanography, production of pure medication, solidification of molten alloys, and geothermally heated lakes and magmas.

There are many investigations available on the effect of time dependent boundary temperature on the onset of Rayleigh–Benard convection. Most of the findings related to this problem have been reviewed by Davis [6]. The bulk of the existing work has concentrated on Rayleigh–Benard convection subject to boundary temperature modulation. A linear stability analysis in case of small amplitude of temperature modulation is performed by Venezian [7]. He has established that the onset of convection can be delayed or advanced by the out of or in phase modulation of the boundary temperatures, respectively as compared to the unmodulated system. It has been found that at low frequencies the equilib-

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Nomenclature

d	height of the fluid layer	m
$\mathbf{g}$	acceleration due to gravity	$m \cdot s^{-2}$
$\mathbf{k}$	unit vector in the vertical direction	m
l, m	wave numbers in x, y directions	m^{-1}
p	pressure	$kg \cdot m^{-1} \cdot s^{-2}$
p_H	basic state pressure	$kg \cdot m^{-1} \cdot s^{-2}$
Pr	Prandtl number, $= \frac{\nu}{\kappa_T}$	
$\mathbf{q}$	velocity vector, (u, v, w)	$m \cdot s^{-1}$
R	thermal Rayleigh number, $= \frac{\beta_1 g \Delta T d^3}{\nu \kappa_T}$	
Rs	solute Rayleigh number, $= \frac{\beta_2 g \Delta S d^3}{\nu \kappa_T}$	
S	solute concentration	$kg \cdot m^{-3}$
S_H	basic state solute concentration	$kg \cdot m^{-3}$
S_R	reference solute concentration	$kg \cdot m^{-3}$
ΔS	salinity difference between the walls	$kg \cdot m^{-3}$
T	temperature	K
T_H	basic state temperature	K
T_R	reference temperature	K
ΔT	temperature difference between the walls	K
t	time	s
(x, y, z)	space coordinates	m

Greek symbols

α	horizontal wave number	
α_c	critical wave number	
β_1	thermal expansion coefficient	K^{-1}
β_2	solute expansion coefficient	$kg^{-1} \cdot m^3$
ε	amplitude of modulation	
κ_S	solute diffusivity	$m^2 \cdot s^{-1}$
κ_T	thermal diffusivity	$m^2 \cdot s^{-1}$
ν	kinematic viscosity	$m^2 \cdot s^{-1}$
τ	diffusivity ratio, $\frac{\kappa_S}{\kappa_T}$	
ρ	density	$kg \cdot m^{-3}$
ρ_H	basic state density	$kg \cdot m^{-3}$
ρ_R	reference density	$kg \cdot m^{-3}$
ϕ	phase angle	
Ω	frequency of the modulation	s^{-1}
ω	non-dimensional frequency, $= \frac{\Omega d^2}{\kappa_T}$	

rium state becomes unstable, because at low frequencies the disturbances grow to a sufficient size that the inertia effects become more important. Rosenblat and Herbert [8], found the asymptotic solution of the low frequency and arbitrary amplitude thermal modulation problem. The solution is discussed from the viewpoint of the stability or otherwise of the basic state, and possible stability criteria are analyzed. They have also made some comparison with known experimental results. Rosenblat and Tanaka [9] have also studied the effect of thermal modulation on the onset of Rayleigh–Benard convection when the temperature gradient has both a steady and time periodic component. It has been found that, in general, there is enhancement of the critical value of a suitably defined Rayleigh number. Roppo et al. [10] have performed weakly nonlinear stability analysis and found that the modulation produces a range of stable hexagons near the critical Rayleigh number. These authors have reported that for low frequencies the modulation is destabilizing, whereas at high frequencies some stabilization is apparent. Finucane and Kelly [11] performed both theoretical and experimental investigation of the thermal modulation in a horizontal fluid layer. A numerical analysis of the linear stability equations indicated that the linear assumption is valid at the low frequencies of modulation. A nonlinear analysis employing the shape assumption and free boundary conditions was developed and examined numerically. They found both experimentally and numerically that at low frequencies the modulation is destabilizing, whereas at high frequencies some stabilization is apparent.

The above mentioned studies have reported that the effect of thermal modulation is to alter the critical value of Rayleigh number by comparison with the unmodulated, steady case. These works are restricted to a single component fluid layer. To our knowledge no studies on the effect of temperature modulation on double diffusion convection in a horizontal two component fluid layer are available in the literature. The purpose of the present paper therefore, is to study the effect of thermal modulation on double diffusive convection in a horizontal fluid layer.

2. Mathematical formulation

We consider a viscous, incompressible two component fluid layer of thickness d and infinite extent in the horizontal direction subject to an adverse temperature gradient and a stabilizing concentration gradient. A Cartesian coordinate system is chosen with the origin in the lower boundary and z-axis vertically upward. With the assumptions and approximations frequently made for the study of double diffusive convection in a horizontal two component fluid layer, the basic equations are:

$$\nabla \cdot \mathbf{q} = 0 \tag{1}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} = -\frac{1}{\rho_R} \nabla p + \frac{\rho}{\rho_R} \mathbf{g} + \nu \nabla^2 \mathbf{q} \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T = \kappa_T \nabla^2 T \tag{3}$$

$$\frac{\partial S}{\partial t} + \mathbf{q} \cdot \nabla S = \kappa_S \nabla^2 S \tag{4}$$

$$\rho = \rho_R \left[1 - \beta_1 (T - T_R) + \beta_2 (S - S_R)\right] \tag{5}$$

The time dependent wall temperatures are externally imposed, and are given by

$$T = T_R + \frac{\Delta T}{2}[1 + \varepsilon \cos \Omega t] \quad \text{at } z = 0 \quad \text{and} \tag{6a}$$

$$T = T_R - \frac{\Delta T}{2}\big[1 - \varepsilon \cos(\Omega t + \phi)\big] \quad \text{at } z = d \tag{6b}$$

Here ε represents a small amplitude, Ω the frequency and ϕ the phase angle.

The boundary conditions on concentration are

$$S = S_R + \frac{\Delta S}{2} \quad \text{at } z = 0 \tag{7a}$$

$$S = S_R - \frac{\Delta S}{2} \quad \text{at } z = d \tag{7b}$$

We consider three types of temperature modulations namely

Case (a) symmetric (in phase, $\phi = 0$),
Case (b) asymmetric (out of phase, $\phi = \pi$), and
Case (c) only lower wall temperature is modulated while the upper one is held at constant temperature ($\phi = -i\infty$).

2.1. Basic state

The basic state of the fluid is quiescent and is described by

$$\mathbf{q} = (0,0,0), \qquad T = T_H(z,t)$$
$$p = p_H(z,t), \qquad \rho = \rho_H(z,t), \qquad S = S_H(z) \tag{8}$$

The temperature $T_H(z,t)$, solute concentration $S_H(z)$, pressure $p_H(z,t)$ and density $\rho_H(z,t)$ satisfy the following equations:

$$\frac{\partial T_H}{\partial t} = \kappa_T \frac{\partial^2 T_H}{\partial z^2} \tag{9}$$

$$\frac{\mathrm{d}^2 S_H}{\mathrm{d}z^2} = 0 \tag{10}$$

$$-\frac{\partial p_H}{\partial z} = \rho_H g \tag{11}$$

$$\rho_H = \rho_R\big[1 - \beta_1(T_H - T_R) + \beta_2(S_H - S_R)\big] \tag{12}$$

The solution of Eq. (9) subject to the boundary conditions (6), consists of the sum of a steady temperature field $T_s(z)$ and an oscillating part $\varepsilon T_1(z,t)$:

$$T_H(z,t) = T_s(z) + \varepsilon T_1(z,t) \tag{13a}$$

where

$$T_s(z) = T_R + \frac{\Delta T}{2}\left(1 - \frac{2z}{d}\right)$$

$$T_1(z,t) = \frac{\Delta T}{2}\big\{Re\big\{\big[a(\lambda)e^{\lambda z/d} + a(-\lambda)e^{-\lambda z/d}\big]e^{-i\Omega t}\big\}\big\}$$

$$\lambda = (1-i)\left[\frac{\Omega d^2}{2\kappa_T}\right]^{1/2}, \qquad a(\lambda) = \left[\frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right]$$

and Re stands for the real part.

The solution of Eq. (10) subject to the boundary conditions (7) is

$$S_H = S_R + \frac{\Delta S}{2}\left(1 - \frac{2z}{d}\right) \tag{13b}$$

2.2. Linear stability analysis

For the small disturbances, we assume solution for $\mathbf{q}$, T, S, p and ρ in the form

$$\mathbf{q} = \mathbf{q}', \qquad T = T_H + T', \qquad S = S_H + S'$$
$$p = p_H + p', \qquad \rho = \rho_H + \rho' \tag{14}$$

where the prime indicates that the quantities are infinitesimal perturbations.

Substituting Eq. (14) into Eqs. (1)–(5) and using the basic state equations and neglecting nonlinear terms in perturbed quantities, we obtain the following equations for the perturbed quantities:

$$\frac{\partial \mathbf{q}'}{\partial t} = -\frac{1}{\rho_R}\nabla p' + \big(\beta_1 T' - \beta_2 S'\big) g\mathbf{k} + \nu\nabla^2\mathbf{q}' \tag{15}$$

$$\frac{\partial T'}{\partial t} + w'\left(\frac{\partial T_H}{\partial z}\right) = \kappa_T \nabla^2 T' \tag{16}$$

$$\frac{\partial S'}{\partial t} + w'\left(\frac{\mathrm{d}S_H}{\mathrm{d}z}\right) = \kappa_S \nabla^2 S' \tag{17}$$

where $\mathbf{k}$ is the unit vector in the positive z direction.

The boundary conditions for the perturbed velocity, temperature and solute concentration are

$$w' = \frac{\partial^2 w'}{\partial z^2} = T' = S' = 0 \quad \text{at } z = 0, d \tag{18}$$

The boundary conditions on velocity are stress-free conditions and those on temperature and solute represent that the boundaries are perfectly conducting to heat and solute.

We eliminate p' from Eq. (15) and render the resulting equation and Eqs. (16) and (17) dimensionless by using the following non-dimensional variables

$$(x,y,z) = (x^*, y^*, z^*)d$$
$$w' = \left(\frac{\kappa_T}{d}\right)w^*, \qquad t = \left(\frac{d^2}{\kappa_T}\right)t^*$$
$$T' = (\Delta T)T^*, \qquad S' = (\Delta S)S^*$$
$$R = \frac{\beta_1 g \Delta T d^3}{\nu\kappa_T}, \qquad Rs = \frac{\beta_2 g \Delta S d^3}{\nu\kappa_T}, \qquad Pr = \frac{\nu}{\kappa_T}$$
$$\tau = \frac{\kappa_S}{\kappa_T}, \qquad \omega = \frac{\Omega d^2}{\kappa_T} \tag{19}$$

where R is the thermal Rayleigh number, Rs is the solute Rayleigh number, Pr is the Prandtl number, τ is the diffusivity ratio and ω is the frequency. The linearized non-dimensional equations are written as (on dropping asterisks):

$$\left(\frac{\partial}{\partial t} - Pr\,\nabla^2\right)\nabla^2 w = R\,Pr\,\nabla_1^2 T - Rs\,Pr\,\nabla_1^2 S \tag{20}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T = -w\frac{\partial T_H}{\partial z} \tag{21}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)S = w \tag{22}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2}$$

Eqs. (20)–(22) are combined to obtain a single differential equation for the vertical component of velocity w as:

$$\begin{aligned}\Big\{&\left(\frac{\partial}{\partial t} - Pr\,\nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\nabla^2 \\ &+ R\,Pr\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\frac{\partial T_H}{\partial z}\nabla_1^2 \\ &+ Rs\,Pr\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla_1^2\Big\}w = 0\end{aligned} \tag{23}$$

The boundary conditions can also be expressed in terms of w by making use of (20), which requires $\frac{\partial^4 w}{\partial z^4} = 0$ at the boundaries. Thus Eq. (23) is to be solved subject to the homogeneous boundary conditions

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = \cdots = 0 \quad \text{at } z = 0, 1 \tag{24}$$

The dimensionless temperature gradient appearing in Eq. (23) can be obtained from Eq. (13a) as

$$\frac{\partial T_H}{\partial z} = -1 + \varepsilon f \tag{25}$$

where

$$f = Re\left\{\left[A(\lambda)e^{\lambda z} + A(-\lambda)e^{-\lambda z}\right]e^{-i\omega t}\right\}$$

$$\lambda = (1-i)\sqrt{\omega/2} \quad \text{and} \quad A(\lambda) = \frac{\lambda}{2}a(\lambda)$$

The horizontal dependence of w is factorable, and we look for solutions with a single wave number α, such that $\nabla_1^2 w = -\alpha^2 w$. The dependence $\exp[-(lx+my)]$ of w on the horizontal coordinates is implied throughout.

3. Method of solution

We apply the perturbation method to obtain the eigenfunctions w and eigenvalues R of Eq. (23) for a temperature profile that departs from the linear one by terms of order ε. Thus it follows that the eigenfunctions and eigenvalues which are obtained in this problem differ from those associated with the two component Rayleigh–Benard problem by quantities of order ε.

Accordingly, we substitute

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots \tag{26}$$

$$R = R_0 + \varepsilon^2 R_2 + \cdots \tag{27}$$

in Eq. (23) and equate coefficients of like powers of ε on both side to obtain (up to order ε^2)

$$L w_0 = 0 \tag{28}$$

$$L w_1 = Pr\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\left\{-R_0 f \nabla_1^2 w_0\right\} \tag{29}$$

$$L w_2 = Pr\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\left\{R_2\nabla_1^2 w_0 - R_0 f \nabla_1^2 w_1\right\} \tag{30}$$

where

$$\begin{aligned}L \equiv &\left(\frac{\partial}{\partial t} - Pr\,\nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\nabla^2 \\ &- R_0 Pr\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\nabla_1^2 + Rs\,Pr\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla_1^2\end{aligned}$$

and w_0, w_1, w_2 are required to satisfy the boundary conditions of Eq. (24).

In Eq. (27) the odd powers of ε are missing because changing the sign of ε shifts the time origin only which does not affect the problem of stability and thus R should be independent of the sign of ε, i.e., R_1, $R_3, \ldots$ must be zero.

The eigenfunction w_0 is the solution of the problem with $\varepsilon = 0$, that is the unmodulated system. The marginally stable solutions for this problem are

$$w_0^{(n)} = \sin n\pi z \tag{31}$$

with corresponding eigenvalues

$$R_0^{(n)} = \frac{(\alpha^2 + n^2\pi^2)^3}{\alpha^2} + \frac{Rs}{\tau} \tag{32}$$

For a fixed value of α the least eigenvalue occurs for $n = 1$. R_0 assumes the minimum value given by

$$R_0 = \left(\frac{27}{4}\pi^4 + \frac{Rs}{\tau}\right) \tag{33a}$$

at $\alpha = \alpha_c$ where α_c is given by

$$\alpha_c = \frac{\pi}{\sqrt{2}} \tag{33b}$$

(For details, see, e.g., [1].)

The equation for w_1 then takes the form

$$L w_1 = R_0 Pr\,\alpha^2\left(\frac{\partial}{\partial t} - \tau\nabla^2\right) f \sin\pi z \tag{34}$$

Now let

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right) = \left(-i\omega + \tau\alpha^2 - \tau D^2\right) \quad \text{where } D = \frac{\mathrm{d}}{\mathrm{d}z}$$

Thus

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \tau\nabla^2\right) f \sin\pi z \\ &\quad = \left[\tau\left(\pi^2 + \alpha^2\right) + i\omega(\tau - 1)\right] f \sin\pi z - 2\tau\lambda\pi f' \cos\pi z\end{aligned} \tag{35}$$

with

$$f' = Re\left\{\left[A(\lambda)e^{\lambda z} - A(-\lambda)e^{-\lambda z}\right]e^{-i\omega t}\right\}$$

Using Eqs. (35), (34) becomes

$$Lw_1 = R_0 Pr\,\alpha^2\,Re\{L_1 f \sin\pi z - 2\pi\tau\lambda f' \cos\pi z\} \tag{36}$$

where

$$L_1 = \tau(\pi^2+\alpha^2) + i\omega(\tau-1)$$

We solve Eq. (36) for w_1 by expanding the right-hand side in Fourier series expansions and inverting the operator L term by term. Thus we obtain

$$w_1 = R_0 Pr\,\alpha^2\left\{L_1\sum\frac{A_n(\lambda)}{L(\omega,n)}\sin n\pi z e^{-i\omega t} - 2\pi\lambda\tau\,Pr\sum\frac{B_n(\lambda)}{L(\omega,n)}\cos n\pi z e^{-i\omega t}\right\} \tag{37}$$

For details, see Appendix A. The equation for w_2 then can be written as

$$Lw_2 = -R_2 Pr\,\tau\alpha^2(\alpha^2+\pi^2)\sin\pi z + R_0 Pr\,\alpha^2 Re\{L_n f w_1 - 2\tau Df\,Dw_1\} \tag{38}$$

We shall not require the solution of this equation, but merely use it to determine R_2, the first non-zero correction to R. The solubility condition requires that the time-independent part of the right-hand side should be orthogonal to $\sin\pi z$. Therefore multiplying Eq. (38) by $\sin\pi z$ and integrating between 0 and 1 we obtain

$$R_2 = \frac{R_0^2\alpha^2}{2\tau(\alpha^2+\pi^2)}Re\left\{\sum_{n=1}^{\infty}\frac{L_1 L_n^*|A_n(\lambda)|^2 L^*(\omega,n)}{|L(\omega,n)|^2} - 4\pi^2|\lambda^2|^2\tau^2\sum_{n=1}^{\infty}\frac{nB_n(\lambda)L^*(\omega,n)C_n^*(\lambda)}{|L(\omega,n)|^2}\right\} \tag{39}$$

For details, see Appendix A. In the above equation the summation extends over even values of n for case (a), odd values of n for case (b) and for all integer values of n for case (c).

The value of the Rayleigh number R obtained by this procedure is the eigenvalue corresponding to the eigenfunction w, which, though oscillating, remains bounded in time. R is a function of the horizontal wave number α and the amplitude of the modulation ε, accordingly we expand

$$R(\alpha,\varepsilon) = R_0(\alpha) + \varepsilon^2 R_2(\alpha) + \cdots \tag{40}$$

$$\alpha = \alpha_0 + \varepsilon^2\alpha_2 + \cdots \tag{41}$$

The critical value of the thermal Rayleigh number R is computed up to $O(\varepsilon^2)$ by evaluating R_0 and R_2 at $\alpha_0 = \alpha_c$ given by Eq. (33b). It is only when one wishes to evaluate R_4, α_2 must be taken in to account [7]. In view of this we write

$$R_c(\alpha,\varepsilon) = R_{0c}(\alpha_0) + \varepsilon^2 R_{2c}(\alpha_0) \tag{42}$$

where R_{0c} and R_{2c} can be obtained from Eqs. (32) and (39), respectively.

If R_{2c} is positive, supercritical instability exists and R has minimum at $\varepsilon = 0$. When R_{2c} is negative, subcritical instability is possible. In this case, we have from Eq. (42), $\varepsilon^2 < (R_{0c}/R_{2c})$. From this we can determine the minimum range of ε, by assigning values to the various physical parameters involved in the problem. Thus the range of the amplitude of modulation, which causes subcritical instabilities in different physical situations, can be explained.

To the order of ε^2, R_{2c} is obtained for the cases where the oscillating temperature field is (a) symmetric, (b) asymmetric and (c) when only the lower wall temperature is oscillating while upper wall is held at constant temperature. The variation of R_{2c} with ω for different values of Pr, τ and Rs are depicted in Figs. 1–6 and the results are discussed in the next section.

4. Results and discussion

The effect of symmetric modulation of the boundary temperatures on the onset of double diffusive convection is shown in Figs. 1–3 for different values of Pr, τ and Rs. We observe that R_{2c} is negative at low frequencies, indicating that the symmetric modulation advances the onset of convection at low frequencies, that is, in the case of symmetric modulation the convection occurs at a lower thermal Rayleigh number than in the unmodulated system. This result is identical to the results obtained by many investigators mentioned in the introduction section in case of single component systems. However for large frequencies, R_{2c} is positive, indicating that the high frequency symmetric modulation has a stabilizing effect. The peak value of R_{2c} depends on the Prandtl number, diffusivity ratio and the solute Rayleigh number.

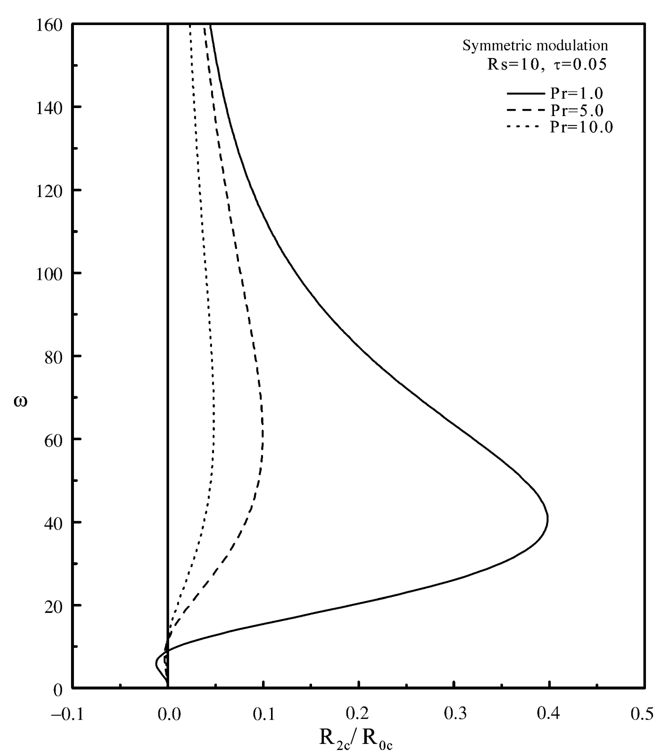

Fig. 1. Variation of R_{2c} with ω for different values of the Prandtl number Pr.

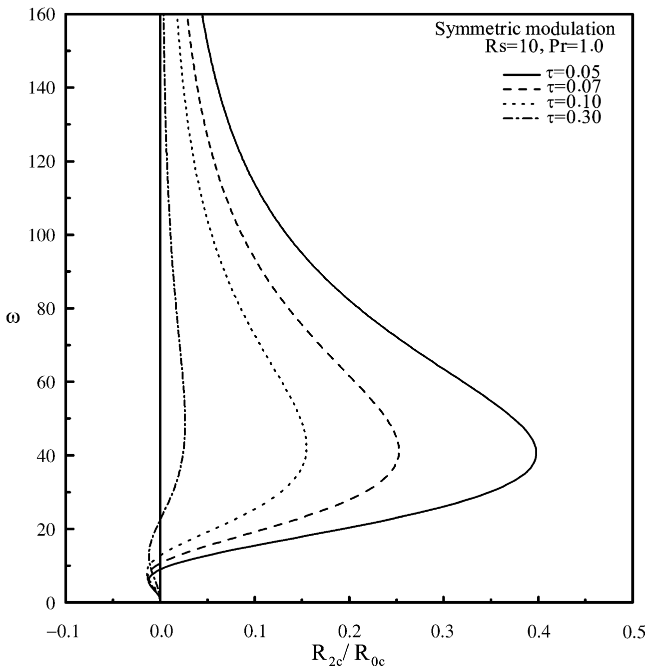

Fig. 2. Variation of R_{2c} with ω for different values of the diffusivity ratio τ.

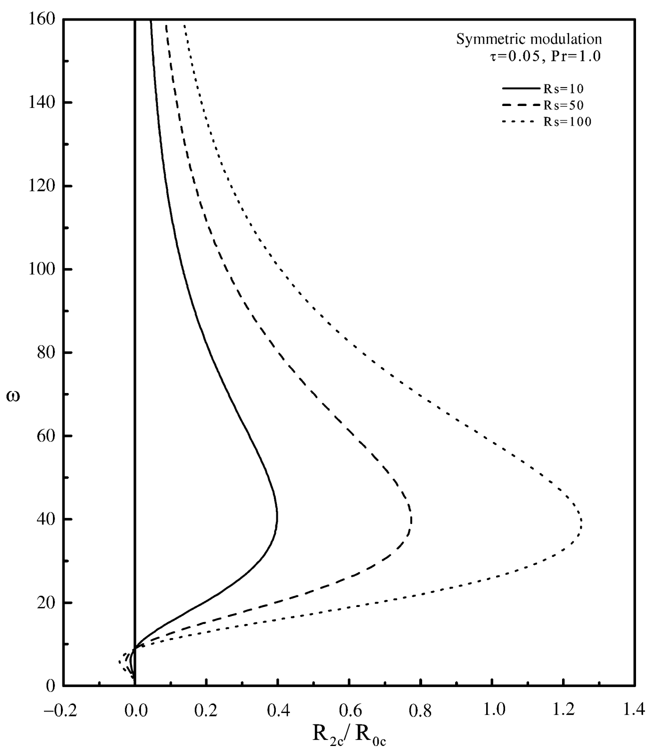

Fig. 3. Variation of R_{2c} with ω for different values of the solute Rayleigh number *Rs*.

Fig. 1 shows the variation of R_{2c} with ω, for different values of Prandtl number *Pr* and fixed values of solute Rayleigh number *Rs* and diffusivity ratio τ, in respect of symmetric modulation of the wall temperature. We find that R_{2c} is positive over a wide range of values of the frequency ω. We also observe from the figure that as Prandtl number *Pr* increases,

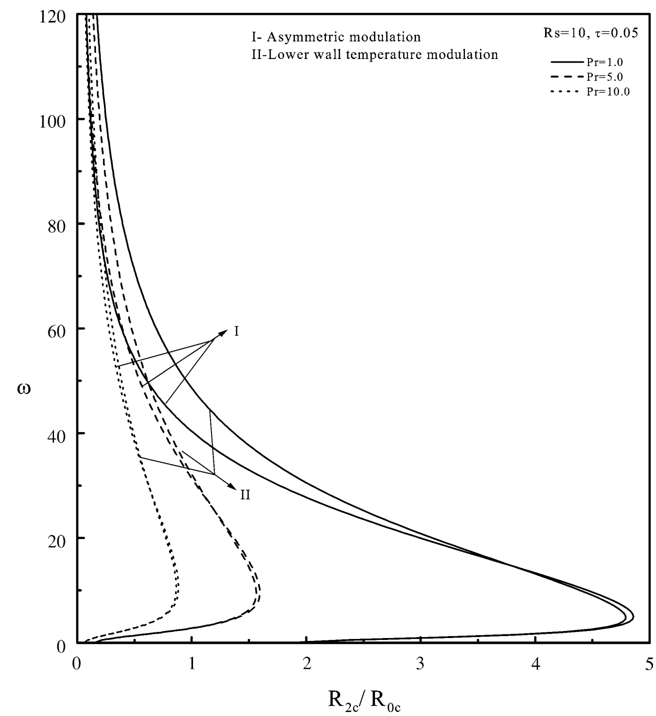

Fig. 4. Variation of R_{2c} with ω for different values of the Prandtl number *Pr*.

the value of R_{2c} decreases indicating that the effect of large Prandtl number is to reduce the effect of the thermal modulation. On the other hand small Prandtl number fluid system is more stable in the presence of thermal modulation as compared to the unmodulated system.

The effect of diffusivity ratio τ on R_{2c} for the case of symmetric modulation of the wall temperature is shown in Fig. 2. We observe that an increase in the value of τ decreases the value of R_{2c}. This indicates that in case of symmetric modulation, the effect of an increase in the value of diffusivity ratio is to minimize the effect of thermal modulation. It is important to note that small values of τ have strong stabilizing effect and on the other hand the effect of modulation is small for larger τ.

The effect of solute Rayleigh number *Rs* on the stability of the system for the case of symmetric modulation of the wall temperature is shown in Fig. 3. We notice that an increase in the value of *Rs* increases the value of R_{2c}, indicating that the effect of large value of the solute Rayleigh number *Rs* is to delay the onset of convection as expected.

Figs. 4–6 illustrates the effect of asymmetric modulation and lower wall temperature modulation on the stability of a two-component fluid system. It is important to note that in these two types of modulation R_{2c} will become positive for the whole range of values of the frequency ω except for $\tau \geqslant 0.3$ in case of lower wall temperature modulation. Thus these two types of modulation have, in general, stabilizing effect. The peak value of R_{2c} depends on the parameters *Rs*, τ and the Prandtl number.

The effect of Prandtl number on the onset of convection in the presence of asymmetric temperature modulation and

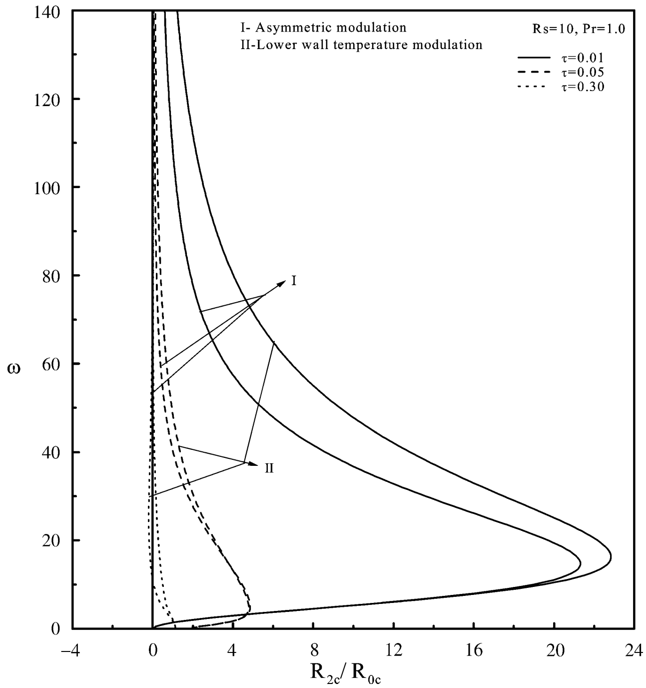

Fig. 5. Variation of R_{2c} with ω for different values of the diffusivity ratio τ.

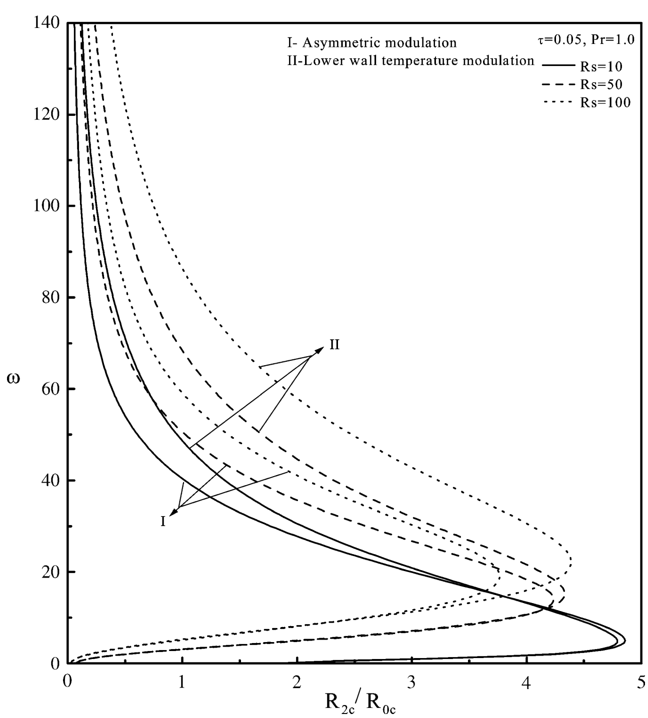

Fig. 6. Variation of R_{2c} with ω for different values of the solute Rayleigh number *Rs*.

lower wall temperature modulation with fixed values of *Rs* and τ is shown in Fig. 4. We note from this figure that the effect of increasing Prandtl number is to minimize the effect of modulation, as in the case of symmetric modulation (Fig. 1). This result is similar to the one in the case of a single component fluid layer with thermal modulation.

Fig. 5 depicts the variation of R_{2c} with ω, for different values of the diffusivity ratio for the case of asymmetric modulation and lower wall temperature modulation. The figure demonstrates that an increase in the value of τ decreases the value of R_{2c}, indicating that, the larger τ reduces the effect of thermal modulation, as in the case of symmetric modulation (Fig. 3). It is interesting to note that R_{2c} become very small for $\tau > 0.3$, for a range of values of ω indicating that for τ larger than 0.3 the system matches the unmodulated case in the presence of asymmetric modulation. In case of lower wall temperature modulation, a range of ω exists where R_{2c} is negative implying a destabilizing effect for $\tau \geqslant 0.3$. These results correspond to the parameter values chosen in the computation.

The effect of solute Rayleigh number *Rs* on correction Rayleigh number R_{2c} for the case of asymmetric modulation and lower wall temperature modulation is shown in Fig. 6. We observe that there is a range of ω where increasing *Rs* decreases R_{2c} and a range of ω where the effect is opposite. The range of ω also depends on other parameters and the type of modulation.

In each case of modulation there is a critical frequency (ω_c) at which the correction Rayleigh number R_{2c} is maximum. It is highly impossible to obtain an explicit analytical relation between this critical frequency and the corresponding correction Rayleigh number. However the critical frequency ω_c at which the correction Rayleigh number R_{2c} is maximum (positive/negative) for different values of Prandtl number, diffusivity ratio and the solute Rayleigh number is reported in Tables 1 and 2. There are two peak values of R_{2c} for the symmetric modulation, one positive and another negative. On the other hand, for the asymmetric and only lower wall temperature modulation cases R_{2c} has only positive peak value except for $\tau \geqslant 0.3$ in the case of lower wall temperature modulation.

The low-frequency thermal modulation has a significant effect on the stability of the system. The results of the present study are expected to be useful in controlling convection by thermal modulation, in a two-component system.

5. Conclusions

In the present paper we made an analytical study of the effect of temperature modulation on double diffusive convection in a two component horizontal fluid layer. The perturbation method is used to find the critical thermal Rayleigh number as a function of frequency of the modulation, Prandtl number, diffusivity ratio and solute Rayleigh number. Three types of thermal modulations are considered and arrived at the following conclusions:

(1) The low frequency symmetric modulation is destabilizing while high frequency symmetric modulation is always stabilizing. The asymmetric modulation and lower wall temperature modulation are, in general, stabilizing

Table 1
Critical frequency ω_c and the corresponding correction Rayleigh number R_{2c} for symmetric modulation

(i) $\tau = 0.050$, $Rs = 10$

Pr	Negative peak value R_{2c}/R_{0c}	ω_c	Positive peak value R_{2c}/R_{0c}	ω_c
1.0	−0.012	6.05	0.39	41
2.0	−0.007	6.55	0.23	50
3.0	−0.005	7.00	0.16	56
4.0	−0.004	7.05	0.12	58
5.0	−0.003	7.05	0.09	60
6.0	−0.003	7.50	0.08	62
7.0	−0.002	7.55	0.07	62
8.0	−0.002	7.55	0.06	62
9.0	−0.002	8.00	0.05	63
10.0	−0.001	8.55	0.04	65

(ii) $Pr = 1.0$, $Rs = 10$

τ	Negative peak value R_{2c}/R_{0c}	ω_c	Positive peak value R_{2c}/R_{0c}	ω_c
0.05	−0.012	6.05	0.398	41.00
0.07	−0.013	7.05	0.252	41.05
0.10	−0.014	8.05	0.154	41.50
0.30	−0.011	13.05	0.026	50.00
0.50	−0.008	13.55	0.007	54.55
0.70	−0.006	13.55	0.002	58.05

(iii) $Pr = 1.0$, $\tau = 0.05$

Rs	Negative peak value R_{2c}/R_{0c}	ω_c	Positive peak value R_{2c}/R_{0c}	ω_c
10	−0.0121	6.05	0.39	41
50	−0.0249	6.05	0.77	41
10^2	−0.0428	6.05	1.24	41
10^3	−0.0012	6.55	4.76	70
10^4	−0.0001	7.00	6.74	99

for all frequencies. However in case of lower wall temperature modulation, there is a range of ω for which the system is unstable when τ is greater than 0.3. Thus, as in the case of a single component system, thermal modulation can destabilize a mode that is stable in the unmodulated case, or stabilize an unstable mode, with the stability characteristics depending on the frequency of the modulation, solute Rayleigh number, Prandtl number and the diffusivity ratio.

(2) The effect of large Prandtl number is found to be destabilizing while small Prandtl number fluid systems are more stable in the presence of thermal modulation. This effect is similar to the one in case of a single component fluid layer.
(3) The effect of increasing solute Rayleigh number is to stabilize the system in the case of symmetric modulation. In case of asymmetric modulation and lower wall temperature modulation, there is a range of ω where increasing Rs decreases R_{2c} and a range of ω where the effect is opposite.
(4) The effect of small diffusivity ratio is to make the system more stable in the presence of thermal modulation.

Table 2
Critical frequency ω_c and the corresponding correction Rayleigh number R_{2c} for asymmetric and lower wall temperature modulation

(i) $\tau = 0.050$, $Rs = 10$

Pr	Asymmetric modulation		Lower wall temperature modulation	
	R_{2c}/R_{0c}	ω_c	R_{2c}/R_{0c}	ω_c
1.0	4.85	5.00	4.78	5.00
2.0	3.14	7.50	3.09	7.05
3.0	2.36	8.55	2.32	8.55
4.0	1.90	9.00	1.86	9.05
5.0	1.59	10.00	1.56	10.00
6.0	1.36	10.00	1.34	10.05
7.0	1.20	10.50	1.18	10.05
8.0	1.07	10.50	1.05	10.50
9.0	0.97	10.55	0.95	10.55
10.0	0.88	11.00	0.86	10.55

(ii) $Pr = 1.0$, $Rs = 10$

τ	Asymmetric modulation		Lower wall temperature modulation			
	R_{2c}/R_{0c}	ω_c	Positive peak value R_{2c}/R_{0c}	ω_c	Negative peak value R_{2c}/R_{0c}	ω_c
0.05	4.85	5.0	4.78	5.00	–	–
0.07	11.9	0.5	11.07	0.55	–	–
0.10	14.8	0.5	13.54	0.55	–	–
0.30	1.27	0.5	1.14	0.50	−0.20	25.55
0.50	0.66	0.05	0.66	0.05	−0.65	20.00
0.70	0.49	0.05	0.49	0.05	−0.99	20.50

(iii) $Pr = 1.0$, $\tau = 0.05$

Rs	Asymmetric modulation		Lower wall temperature modulation	
	R_{2c}/R_{0c}	ω_c	R_{2c}/R_{0c}	ω_c
10	4.85	05	4.78	5.00
50	4.23	14	4.33	15.0
10^2	3.76	19	4.38	22.0
10^3	2.36	60	7.37	65.0
10^4	11.37	64	16.37	79.0

(5) The effect of modulation disappears for large frequencies, so that the stability or instability of the system matches the unmodulated case, as in the case of a single component system.

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Appendix A

We solve Eq. (36) for w_1 by expanding the right-hand side of it in Fourier series expansions and inverting the operator L term by term. For this, we need the following Fourier series expansions:

$$g_{nm}(\lambda) = 2\int_0^1 e^{\lambda z}\sin n\pi z \sin m\pi z \,\mathrm{d}z = -\frac{4nm\pi^2\lambda[1+(-1)^{n+m+1}e^{\lambda}]}{[\lambda^2+(n+m)^2\pi^2][\lambda^2+(n-m)^2\pi^2]} \quad (A.1)$$

$$f_{nm}(\lambda) = 2\int_0^1 e^{\lambda z}\cos n\pi z \cos m\pi z \,\mathrm{d}z = -\frac{2\lambda[\lambda^2+(n^2+m^2)\pi^2][1+(-1)^{n+m+1}e^{\lambda}]}{[\lambda^2+(n+m)^2\pi^2][\lambda^2+(n-m)^2\pi^2]} \quad (A.2)$$

so that

$$e^{\lambda z}\sin m\pi z = \sum_{n=1}^{\infty} g_{nm}\sin n\pi z \quad (A.3)$$

$$e^{\lambda z}\cos m\pi z = \sum_{n=1}^{\infty} f_{nm}\cos n\pi z \quad (A.4)$$

Let us now define

$$L(\omega,n) = \left(\omega^2 B_1 - B_3\right) - i\omega B_2 \quad (A.5)$$

where

$$B_1 = \left(\alpha^2+n^2\pi^2\right)^2\left(\tau+(1+Pr)\right)$$
$$B_2 = \omega^2\left(\alpha^2+n^2\pi^2\right) - \left(\alpha^2+n^2\pi^2\right)^3\left(Pr+\tau(1+Pr)\right) + Pr\,\alpha^2(R_0-Rs)$$
$$B_3 = \tau\,Pr\left(\alpha^2+n^2\pi^2\right)^4 - Pr\,\alpha^2\left(\alpha^2+n^2\pi^2\right)(\tau R_0 - Rs)$$

It is easily seen that

$$L\left(\sin n\pi z e^{-i\omega t}\right) = L(\omega,n)\sin n\pi z e^{-i\omega t}$$
$$L\left(\cos n\pi z e^{-i\omega t}\right) = L(\omega,n)\cos n\pi z e^{-i\omega t}$$

and Eq. (36) now reads

$$Lw_1 = R_0 Pr\,\alpha^2 Re\Big\{\sum L_1\big[A(\lambda)g_{n1}(\lambda) + A(-\lambda)g_{n1}(-\lambda)\big]\sin n\pi z e^{-i\omega t} - 2\pi\lambda\tau\sum\big[A(\lambda)f_{n1}(\lambda) + A(-\lambda)f_{n1}(-\lambda)\big]\cos n\pi z e^{-i\omega t}\Big\} \quad (A.6)$$

so that

$$w_1 = R_0 Pr\,\alpha^2\left\{L_1\sum\frac{A_n(\lambda)}{L(\omega,n)}\sin n\pi z e^{-i\omega t} - 2\pi\lambda\tau\,Pr\sum\frac{B_n(\lambda)}{L(\omega,n)}\cos n\pi z e^{-i\omega t}\right\} \quad (A.7)$$

where

$$A_n(\lambda) = A(\lambda)g_{n1}(\lambda) + A(-\lambda)g_{n1}(-\lambda)$$
$$B_n(\lambda) = A(\lambda)f_{n1}(\lambda) - A(-\lambda)f_{n1}(-\lambda)$$

To simplify Eq. (30) for w_2, we need

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right) f w_1 = L_n f w_1 - 2\tau Df\,Dw_1 \quad (A.8)$$

where

$$L_n = \tau\left(\alpha^2+n^2\pi^2\right) + i\omega(\tau-1)$$

The equation for w_2 then can be written as

$$Lw_2 = -R_2 Pr\,\tau\alpha^2\left(\alpha^2+\pi^2\right)\sin\pi z + R_0 Pr\,\alpha^2 Re\{L_n f w_1 - 2\tau Df\,Dw_1\} \quad (A.9)$$

The solubility condition requires that the time-independent part of the right-hand side of Eq. (A.9) must be orthogonal to $\sin\pi z$. Multiply Eq. (A9) by $\sin\pi z$ and integrating between 0 and 1 we obtain

$$R_2 = \left[\frac{2R_0}{\tau(\alpha^2+\pi^2)}\right] Re\left\{L_n\int_0^1 \overline{f w_1}\sin\pi z\,\mathrm{d}z - 2\tau\int_0^1 \overline{Df\,Dw_1}\sin\pi z\,\mathrm{d}z\right\} \quad (A.10)$$

where an over bar denotes the time average.

We have the Fourier series expansions

$$f\sin\pi z = Re\sum A_n(\lambda)\sin n\pi z e^{-i\omega t} \quad (A.11)$$

$$Df\sin\pi z = Re\sum \lambda C_n(\lambda)\sin n\pi z e^{-i\omega t} \quad (A.12)$$

where

$$C_n(\lambda) = A(\lambda)g_{n1}(\lambda) - A(-\lambda)g_{n1}(-\lambda)$$

We also note the fact that the time average of product of two complex functions A and B is given by

$$\overline{A\cdot B} = \frac{1}{2\pi}\int_0^{2\pi} AB\,\mathrm{d}t = \frac{1}{2}A^*B = \frac{1}{2}AB^* \quad (A.13)$$

where $*$ denotes a complex conjugate.

Using Eqs. (A.12) and (A.13) in Eq. (A.11) we obtain

$$R_2 = \frac{R_0^2\alpha^2}{2\tau(\alpha^2+\pi^2)} Re\left\{\sum_{n=1}^{\infty}\frac{L_1 L_n^*|A_n(\lambda)|^2 L^*(\omega,n)}{|L(\omega,n)|^2} - 4\pi^2\left|\lambda^2\right|^2\tau^2\sum_{n=1}^{\infty}\frac{nB_n(\lambda)L^*(\omega,n)C_n^*(\lambda)}{|L(\omega,n)|^2}\right\} \quad (A.14)$$

Eq. (A.9) can now be solved for w_2, and the procedure may be continued to obtain further corrections to w and R.

We need the real part of $(L_1 L_n^* L^*)$ which can be easily calculated

$$Re\left\{L_1 L_n^* L^*\right\} = \left(\omega^2 B_1 - B_3\right)B_4 + \omega B_2 B_5 \quad (A.15)$$

$$\left|L(\omega,n)\right|^2 = \left(\omega^2 B_1 - B_3\right)^2 + \omega^2 B_2^2 \quad (A.16)$$

$$\left|A_n(\lambda)\right|^2 = \frac{16\pi^4 n^2\omega^2}{[\omega^2+(n+1)^4\pi^4][\omega^2+(n-1)^4\pi^4]} \quad (A.17)$$

where

$$B_4 = \tau^2\left(\alpha^2 + \pi^2\right)\left(\alpha^2 + n^2\pi^2\right) + \omega^2(\tau - 1)^2$$

$$B_5 = \omega\tau(\tau - 1)\left(\left(\alpha^2 + \pi^2\right) - \left(\alpha^2 + n^2\pi^2\right)\right)$$

Similarly we can also find real part of $(B_n(\lambda)L^*(\omega, n)C_n(\lambda))$ easily.

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